Partial Dual and Delta-Matrix

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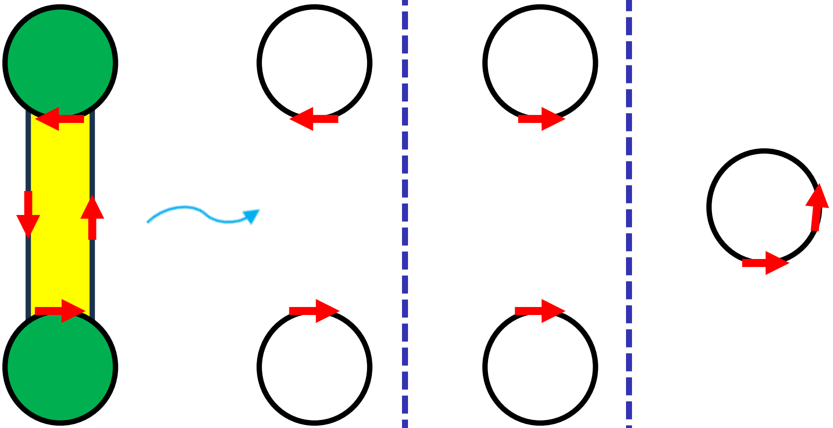
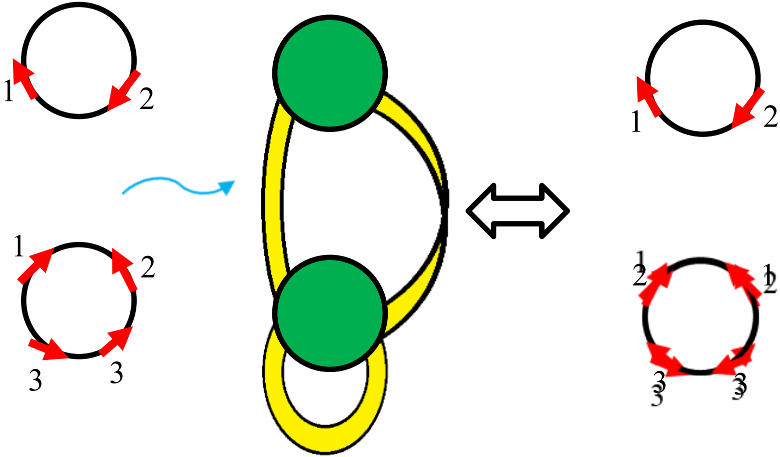
June 27, 2025

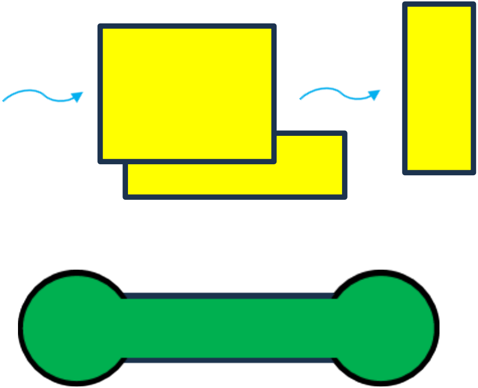
A more combinatorial definition of ribbon graphs



A more combinatorial definition of ribbon graphs



The table of Partial Dual



$G = (V,E); A \subseteq E; G^{\delta(A)}$. $e \in E, G^{\delta(e)}$ $\{e\} \subseteq E, G^{\delta(\{e\})}$

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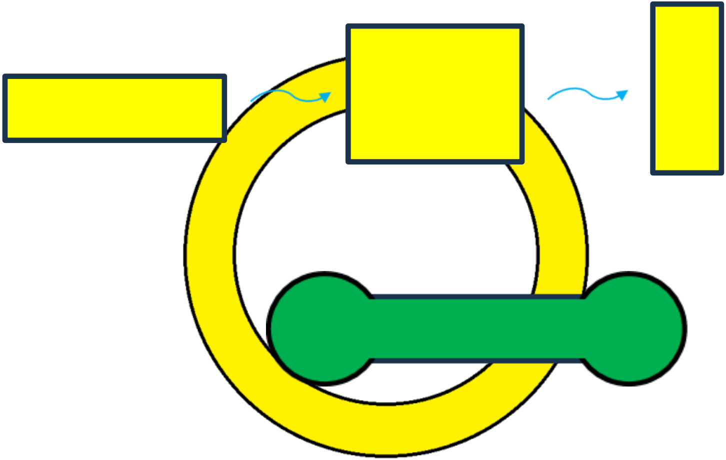

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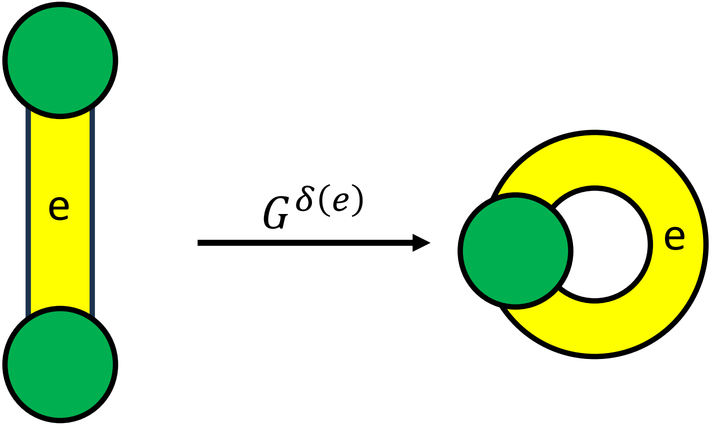
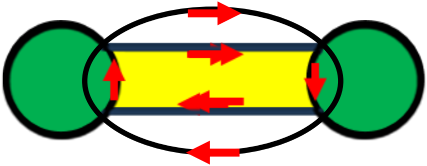


The table of Partial Dual



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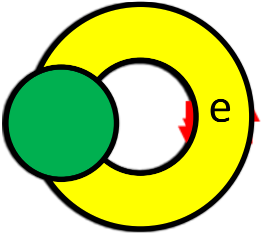
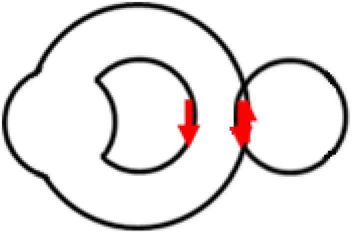
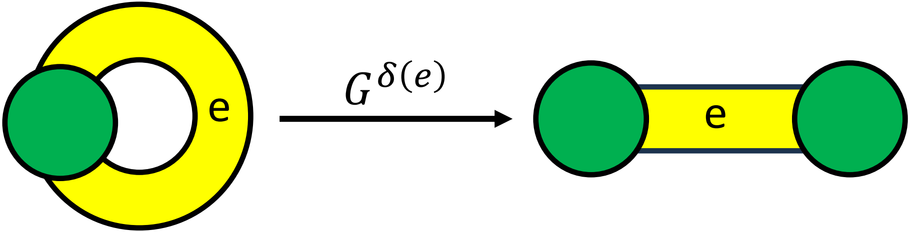


The table of Partial Dual



The table of Partial Dual



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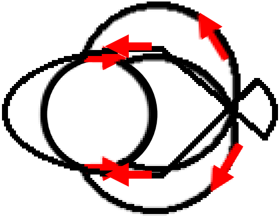
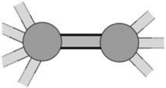
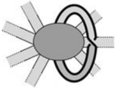
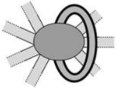
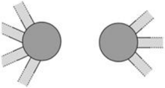
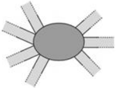
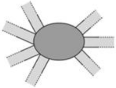
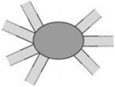
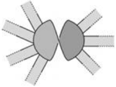
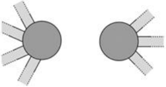
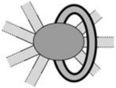
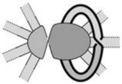
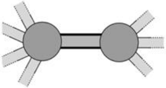


The table of Partial Dual

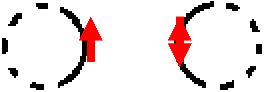
	Non-loop	Non-orientable loop	Orientable loop
G			
$G\backslash e$			
$\boxed{G/e = G^{\delta(e)}\backslash e}$			
$G^{\delta(e)}$			

$$\boxed{G^{\delta(A)}}$$

Chun, C. and Moffatt, I. and Noble, Steven and Rueckriemen, R. (2018) On the interplay between embedded graphs and delta-matroids. Proceedings of the London Mathematical Society 118 (3), pp. 675-700. ISSN 0024-6115.

Partial Petrial

$$G = (V, E); A \subseteq E; G^{\tau(A)}. \{\boldsymbol{e}\} \in \subseteq E E, ; G G^{\tau(\boldsymbol{e}()\{e\})}$$



Some Lemmas

(1) Suppose that an edge e does not belong to A,
then $G^{\delta(A\cup\{e\})} = \left(G^{\delta(A)}\right)^{\delta(e)}$

(2) $G = \left(G^{\delta(A)}\right)^{\delta(A)}$

(3) $\left(G^{\delta(A)}\right)^{\delta(B)} = G^{\delta(A\triangle B)}$

(4) $(G/e)^{\delta(A)} = G^{\delta(A\cup\{e\})}\backslash e$

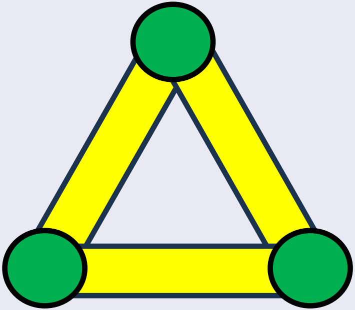
(5) $(G\backslash e)^{\delta(A)} = G^{\delta(A\cup\{e\})}/e = G^{\delta(A)}\backslash e$

(6) $G = \left(G^{\tau(A)}\right)^{\tau(A)}$

Spanning subgraph

Definition:
Let $G = (V, E)$ be a ribbon graph. G' is a Spanning subgraph of G iff $G' = (V, E')$ with some $E' \subseteq E$.

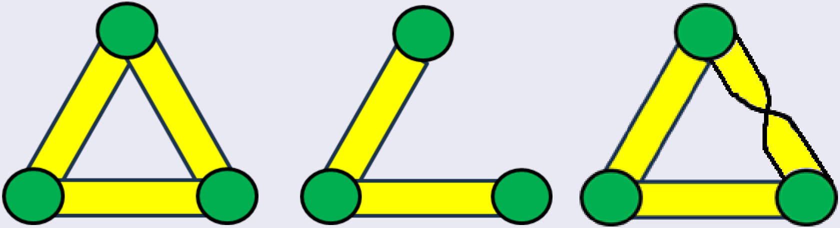
Examples:



Spanning Quasi-tree

Definition:
Let $G = (V, E)$ be a ribbon graph, and let G' be a Spanning subgraph of G. If G' has exactly one boundary component, then G' is a *spanning quasi-tree* of G

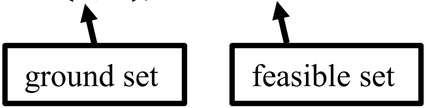
Examples:



Delta-Matroid

A set system:

$D = (E, \mathcal{F})$, where $\forall F \in \mathcal{F}, F \subseteq E$



ground set

feasible set

Definition:
A *delta-matroid* is a proper set system $D = (E, \mathcal{F})$ that satisfies the Symmetric Exchange Axiom

Symmetric Exchange Axiom:
For $X, Y \in \mathcal{F}$; for $\forall u \in X \triangle Y, \exists v \in X \triangle Y, \mathrm{s.t.}\, X \triangle \{u, v\} \in \mathcal{F}$

Delta-Matroid

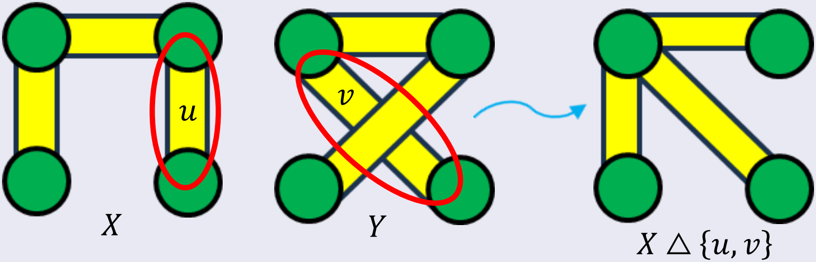
Definition:

Let $G = (V, E)$ be a ribbon graph. And let

$\mathcal{F} := \{F \subseteq E | F \text{ is the edge set of a spanning quasi−tree of } G\}$

We call $D(G) = (E, \mathcal{F})$ the *delta-matroid* of G.

Verification:



Twist of Delta-Matroids

Definition:
Let $D = (E, \mathcal{F})$ be a set system. For $A \subseteq E$, the twist of D with respect to A, denoted by $D * A$, is given by

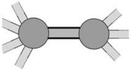
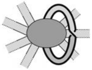
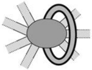
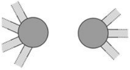
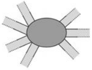
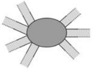
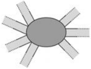
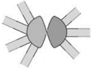
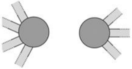
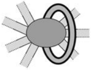
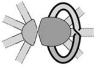
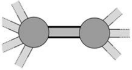
$$D * A = (E, \{A \triangle X | X \in \mathcal{F}\})$$

An important proposition

Proposition:
Let $G = (V, E)$ be a ribbon graph,and $A \subseteq E$, then

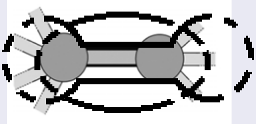
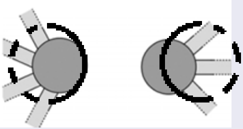
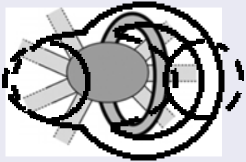
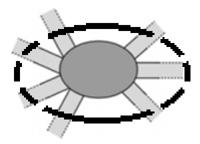
$$D(G) * A = D\left(G^{\delta(A)}\right)$$

Proof:

	Non-loop	Non-orientable loop	Orientable loop
G			
$G\backslash e$			
$G/e = G^{\delta(e)}\backslash e$			
$G^{\delta(e)}$			

An important proposition

Proof (continued):

	$e \in F$	$e \notin F$
G		
$G^{\delta(e)}$		

An important proposition

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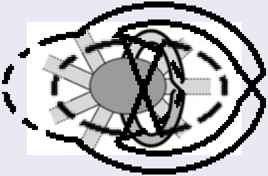
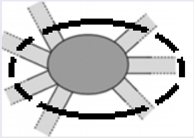
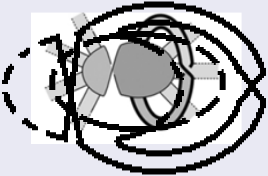
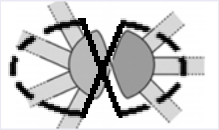
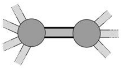
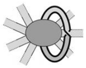
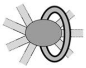
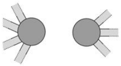
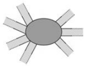
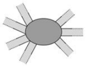
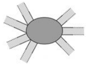
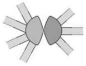
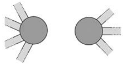
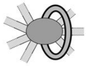
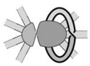
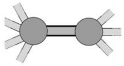
G		
$G^{\delta(e)}$		

An important proposition

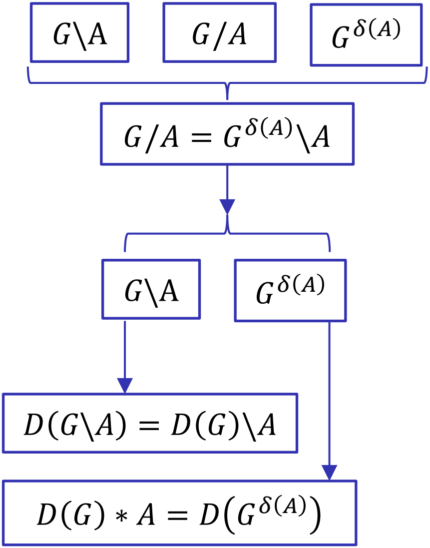
Proof (continued):

	Non-loop	Non-orientable loop	Orientable loop
G			
$G \backslash e$			
$G/e = G^{\delta(e)} \backslash e$			
$G^{\delta(e)}$			

(1) Suppose that an edge e does not belong to A, then $G^{\delta(A \cup \{e\})} = \left(G^{\delta(A)}\right)^{\delta(e)}$

Q.E.D

A possible use of this proposition



$G\backslash A$

G/A

$G^{\delta(A)}$

$G/A = G^{\delta(A)}\backslash A$

$G\backslash A$

$G^{\delta(A)}$

$D(G\backslash A) = D(G)\backslash A$

$D(G) * A = D\left(G^{\delta(A)}\right)$

$$\kappa(G) = \kappa(G\backslash e) + \kappa(G/e)$$

$$\Updownarrow$$

$$\begin{aligned}&\left|\mathcal{F}\left(D(G)\right)\right| \\ &= \left|\mathcal{F}\left(D(G\backslash e)\right)\right| + \left|\mathcal{F}\left(D(G/e)\right)\right|\end{aligned}$$